Journal of Engineering Physics and Thermophysics, Vol. 79, No. 2, 2006

ASYMPTOTIC SCREENING AND NETWORK MODELS FOR CLOSELY PACKED PARTICLES

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UDC 519.6:517.58

The asymptotics of the conductivity (capacitance) of a medium with a system of perfectly conductive particles with a small spacing between them (close packing of particles) has been investigated. A sufficient condition for the appearance of the asymptotic screening effect has been obtained. It has been shown that for this effect to be present for pairs of neighboring particles the continuous problem can be approximated by a finite-dimensional (network) problem.

Consider the boundary-value problem for the Laplace equation in a region with perfectly conductive inclusions on which the solution acquires a constant (but unknown) value. One method for analyzing such problems is rigid expansion of the problem [1], but it is unsuitable at small spacings between inclusions. The other approaches proposed in [2, 3] that are based on the reduction of the initial continuous problem to a network model are essentially two-dimensional, and the method presented in [2] is fit only for problems with smooth coefficients (which does not occur in problems of material science). The approach proposed in [3] was developed for mixtures. No other mathematically justified network models are known to the author.

A solution to the problem on the electric field in a system of periodic bodies was first proposed in [4]. Under the condition of a small spacing δ between bodies, the problem was considered in [5], where the inapplicability of the formulas of [4] for closely packed particles was shown and new computing formulas were obtained. In [6], the screening effect at an approach of bodies is described (by the screening is meant not the enclosure of bodies in a closed conductor–screen, but the independence of capacitance of a pair of close bodies of the presence of other bodies). In the last few years, a large number of works have appeared (see the references in [3, 7]), where finite-dimensional (network) models of continuous problems are used.

The aim of the present paper is to show that the question of the existence of finite-dimensional (network) models for continuous problems and the screening effect are interrelated and, what is more, its presence makes it possible to use network models (i.e., they are "secondary" to the screening effect).

As will be shown below, the screening effect described in [6] is not always observed. A sufficient condition for its appearance is the tending of the pair capacitance of bodies to infinity as they draw closer together. In the three-dimensional case, the pair capacitance tends to infinity (and, therefore, network models are applicable), e.g., for a sphere–sphere pair, but not for a cone–plane pair. Note that these pairs exhaust the practically possible local geometries of particles.

In the two-dimensional case, the capacitance of a pair of bodies increases for both a disk–disk pair and an angle–disk pair. Thus, the possibilities of using network models in the two-dimensional and three-dimensional cases radically differ.

In practice, two kinds of particles occur: smooth particles (obtained, e.g., by the technology of producing pellets) and multifaceted ones (obtained by crushing, as a rule, by milling).

Problem Formulation. Consider a rectangle $\Pi = [-L, L]^3$, in which convex particles D_i, $i = 1, 2, ..., N$, with sectionally smooth boundaries are distributed (Fig. 1). Denote the region outside of the particles by $Q = \Pi \setminus \bigcup_{i=1}^{N} D_i$. Formulate the problem

$$\Delta\varphi = 0 \quad \text{in} \quad Q\,; \tag{1}$$

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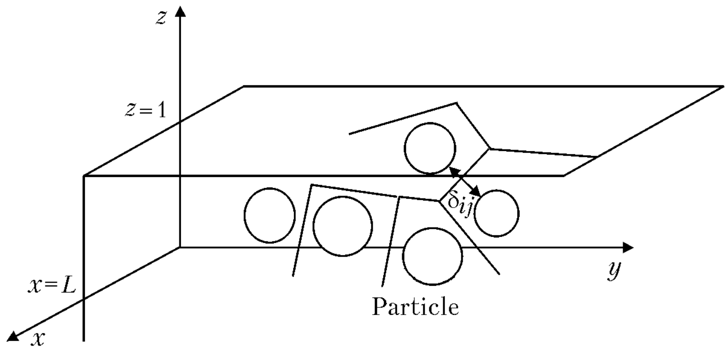

Fig. 1. Scheme of the composite.

$$\varphi = t_i \quad \text{on} \quad D_i\,, \quad i = 1, 2, ..., N\,; \tag{2}$$

$$\int\limits_{\partial D_i} \frac{\partial \varphi}{\partial \mathbf{n}}\, d\mathbf{x} = 0\,, \quad i = 1, 2, ..., N\,; \tag{3}$$

$$\varphi\,(x, y, \pm 1) = \pm 1\,; \tag{4}$$

$$\frac{\partial \varphi}{\partial \mathbf{n}}(\pm L, y, z) = 0\,, \quad \frac{\partial \varphi}{\partial \mathbf{n}}(x, \pm L, z) = 0\,. \tag{5}$$

The unknowns here are the functions $\varphi(\mathbf{x})$ in the region Q and its values (t_i) on D_i particles. Problem (1)–(5) describes, in particular, a capacitor with plates $z = \pm 1$, between which a system of conductors D_i is placed.

Note that while mathematically the perfect conductance condition is traditionally formulated as $\varphi = t_i$ on ∂D_i (as the boundary condition), physically it is realized in the form of (2) as the (electric, thermal, etc.) field constancy condition in a perfectly conducting inclusion D_i. The possibility in (2) "to make no difference" between ∂D_i and D_i corresponds to the "physics" of the problem. Condition (3) is obtained by integrating by parts the Euler equation for the functional $I(\varphi)$, taking into account the condition $\varphi(\mathbf{x}) = t_i$ on D_i. The boundary condition (4) implies the application on the horizontal faces $z = \pm 1$ of the region P of potentials (voltages, temperatures, etc.) equal to ± 1, respectively, and (5) — the absence of a flux through the vertical faces of the region Π.

The boundary condition (1)–(5) is equivalent to the minimization problem

$$I\,(\varphi) = \frac{1}{2}\int\limits_{Q} |\nabla \varphi|^2\, d\mathbf{x} \to \min \tag{6}$$

on the set of functions

$$V = \left\{ \varphi\,(\mathbf{x}) \in H^1\,(Q) : \ \varphi\,(\mathbf{x}) = t_i \quad \text{on} \quad D_i\,, \quad \varphi\,(x, y, \pm 1) = \pm 1 \right\}. \tag{7}$$

The dual extreme principle is of the form [3]

$$J\,(\mathbf{v}) = \left(\int\limits_{Q} -\frac{1}{2}\mathbf{v}^2\, d\mathbf{x} + \int\limits_{z=+1} \varphi^0\, \mathbf{v}\mathbf{n} d\mathbf{x} \right) \to \max \tag{8}$$

on the set of functions

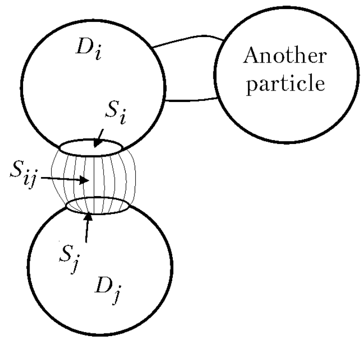

Fig. 2. Layout of the particles and the channel.

$$W = \Big\{ \mathbf{v} = (v_1(\mathbf{x}),\ v_2(\mathbf{x}),\ v_3(\mathbf{x})) \in L_2(Q):\ \mathrm{div}\,\mathbf{v} \in L_2(Q),\ \mathbf{v}(\pm L, y, z)\,\mathbf{n} = 0,$$

$$\int\limits_{\partial D_i} \mathbf{vn}d\mathbf{x} = 0,\ i = 1, 2, ..., N \Big\}, \tag{9}$$

satisfying the condition div $\mathbf{v}$ = 0. In (8), $\varphi^0(z)$ denotes the function for which $\varphi^0(\pm 1) = \pm 1$ (used exclusively to shorten the writing of formulas).

The effective conductance of an inhomogeneous medium is physically defined as a specific flux through a boundary, e.g., the upper boundary (given by the condition $z = 1$):

$$a = \frac{1}{2L} \int\limits_{z=1} \frac{\partial \varphi}{\partial \mathbf{n}} d\mathbf{x}. \tag{10}$$

To make use of extreme principles, one has to know the expression of the flux through the functional $I(\varphi)$. To obtain the corresponding formula, multiply (1) by $\varphi \in V$ and integrate the result by parts. Manipulations (see [3]) yield

$$\int\limits_{z=1} \frac{\partial \varphi}{\partial \mathbf{n}} d\mathbf{x} = \frac{1}{2} \int\limits_{Q} |\nabla \varphi|^2 d\mathbf{x}, \tag{11}$$

where $\varphi(\mathbf{x})$ is the solution of the boundary problem (1)–(5) or, what is the same, of problems (6), (7) or (8), (9).

Further, it will be convenient to use the quantity $A = 2La$ (total normal flux through the boundary $z = 1$). By virtue of (10), (11),

$$A = \frac{1}{2} \int\limits_{Q} |\nabla \varphi|^2 d\mathbf{x}. \tag{12}$$

Capacitance to the Channel. In the composite represented in Fig. 1, let us introduce channels between neighboring particles. A particle can have several neighbors. Therefore, the channel S_{ij} (Fig. 2) should not be too wide to avoid crossing other channels. At the same time, its width is fixed.

Consider the problem on the electric field created by two particles D_i and D_j in R^3:

$$\Delta \varphi = 0 \quad \text{in} \quad R^3 \setminus (D_i \cup D_j); \tag{13}$$

$$\varphi = t_i \quad \text{on} \quad D_i; \quad \varphi = t_j \quad \text{on} \quad D_j;$$

$$|\varphi(\mathbf{x})| \to 0 \quad \text{at} \quad |\mathbf{x}| \to \infty,$$

where t_i and t_j are given numbers. It has a unique solution. For a fairly large number of regions, the solutions of (13) have been obtained explicitly or their properties are known.

Denote the solution of (13) at $t_i = -1/2$ and $t_j = 1/2$ as $\varphi^{\pm 1}(\mathbf{x})$. Apparently, $\nabla\varphi = \nabla\varphi^{\pm 1}(t_i - t_j)^2$.

We will call the quantity

$$C^{S_{ij}} = \int\limits_{S_{ij}} |\nabla\varphi^{\pm 1}|^2 \, d\mathbf{x} = \int\limits_{S_j} \nabla\varphi^{\pm 1} \, \mathbf{n} d\mathbf{x} \tag{14}$$

the capacitance of the sets S_i and S_j (or D_i and D_j) to the set S_{ij}. Here S_i is the part of the boundary ∂D_i adjoining the channel S_{ij} (Fig. 2); $C^{S_{ij}}$ is the capacitance of the capacitor with plates S_i and S_j under the condition that the electric field does not fall outside the limits of the region S_{ij}. Although physically this is impossible (the field is present throughout), the mathematical definition is correct.

The quantity

$$C_{ij}^{(2)} = \int\limits_{R^n(D_i \cup D_j)} |\nabla\varphi^{\pm 1}|^2 \, d\mathbf{x} \tag{15}$$

is the capacitance of the sets D_i and D_j in R^n, $n = 2, 3$ (i.e., classical pair capacitance of bodies D_i and D_j). For a fairly large number of regions, their pair capacitances in R^2 and R^3 can be calculated explicitly [8].

Asymptotic Screening. The effect of asymptotic screening (as described in [6]) consists of the absence of the influence (possibility of ignoring the influence) of other bodies on the mutual capacitance of two close bodies [6]. It turns out, however (which will be shown below), that the screening effect is either observed or absent. This depends on the particle shape.

Let two particles D_i and D_j exist in R^3. Denote the spacing between them by δ. Separate the channel S_{ij} connecting the particle.

Screening lemma. *For the above particles when* $\delta \to 0$:

(1) *the energy outside the channel*

$$\int\limits_{R^3 \setminus (S_{ij} \cup D_i \cup D_j)} |\nabla\varphi|^2 \, d\mathbf{x} \le C,$$

where the constant $C < \infty$ *can be chosen to be independent of* δ;

(2) *if for any neighboring particles* D_i and D_j *the condition* $C_{ij}^{(2)} \to \infty$ *when* $\delta \to 0$ *is met, then capacitances* $C_{ij}^{(2)}$ *and* $C^{S_{ij}}$ *are asymptotically equivalent*: $C_{ij}^{(2)} \sim C^{S_{ij}}$;

(3) *the energy in the* S_{ij} *channel is equal to* $\frac{1}{2} C^{S_{ij}}(t_i - t_j)^2 \sim \frac{1}{2} C_{ij}^{(2)}(t_i - t_j)^2$, *where* t_i *and* t_j *are the potentials of particles* D_i and D_j.

We will write $f \sim g$ when $\delta \to 0$ if $\frac{f}{g} \to 1$ when $\delta \to 0$. The proof of the lemma is mathematical in nature and is not given because of its awkwardness. We will only give the physical justification.

The regions outside the channel are separated from the particles D_i and D_j by a finite distance even if the particles are very close. Consequently, the potential φ gradient (field strength $\nabla\varphi$) outside D_i and D_j is uniformly δ-finite. At infinity, the potential attenuates so that the energy is uniformly δ-bounded [6], which follows from item (1) of the lemma. Mathematical justification of this statement can be carried out on the basis of estimates of the Douglas–Nirenberg type [9] for derivatives of the Laplace equation inside the regions and near the smooth boundaries.

The asymptotic equivalence $C_{ij}^{(2)} \sim C^{S_{ij}}$ of item (2) of the lemma follows from the equality

$$C_{ij}^{(2)} = C^{S_{ij}} + \int\limits_{R^3 \setminus (S_{ij} \cup D_i \cup D_j)} |\nabla\varphi^{\pm 1}|^2 \, d\mathbf{x}, \tag{16}$$

following from the definition of the quantities $C^{S_{ij}}$ and $C_{ij}^{(2)}$ (14), (15). And the integral in (16) thereby is uniformly δ-bounded by virtue of item (1) of the lemma. If

$$\int_{R^3\backslash(D_i\cup D_j)} |\nabla\varphi^{\pm 1}|^2\, d\mathbf{x} \to \infty \quad \text{when} \quad \delta \to 0\,,$$

then $C^{S_{ij}} \to \infty$ when $\delta \to 0$ by virtue of the uniform δ-boundedness of $\int_{R^3\backslash(S_{ij}\cup D_i\cup D_j)} \left|\nabla\varphi^{\pm 1}\right|^2 d\mathbf{x}$ (item (1) of the lemma). Dividing (16) by $C^{S_{ij}}$, we obtain

$$C_{ij}^{(2)}/C^{S_{ij}} = 1 + \int_{R^3\backslash(S_{ij}\cup D_i\cup D_j)} |\nabla\varphi^{\pm 1}|^2\, d\mathbf{x}/C^{S_{ij}} \to 1 \quad \text{at} \quad \delta \to 0\,,$$

since the integral is uniformly δ-bounded, and $C^{S_{ij}} \to \infty$ when $\delta \to 0$.

A condition sufficient for the appearance of the effect of asymptotic screening for neighboring particles $C_{ij}^{(2)} \to \infty$ when $\delta \to 0$ has been obtained.

In so doing, by item (1) of the lemma, it has been established that the energy is concentrated in the S_{ij} channels between particles. The energy in the S_{ij} channel is asymptotically (when $\delta \to 0$) equal to $\frac{1}{2} C_{ij}^{(2)}(t_i - t_j)^2$.

Approximation of the Continuous Problem by the Discrete Problem for Screening Particles. Let us construct a network model corresponding to problem (1)–(5) and give its physical justification. From the screening lemma it follows that if $C_{ij}^{(2)} \to \infty$ when $\delta \to 0$, then the energy is concentrated in the channels, and in the S_{ij} channel it is asymptotically equal to $\frac{1}{2} C_{ij}^{(2)}(t_i - t_j)^2$. Then the flux in the S_{ij} channel is equal to $C_{ij}^{(2)}(t_i - t_j)$. Let us construct a network (graph) $\left\{x_i, t_i, C_{ij}^{(2)}, i, j = 1, 2, ..., N\right\}$, where x_i denotes network nodes (particles); t_i is the particle potential; the capacitances $C_{ij}^{(2)}$ are the characteristics of the network edges.

By virtue of condition (3), the flows in the network satisfy the Kirchhoff equation

$$\sum_{j=1}^{N} C_{ij}^{(2)}\,(t_i - t_j) = 0\,,\ \ i \in I \tag{17}$$

for internal network nodes (hereinafter denoted by I) and the boundary conditions

$$t_i = \pm\, 1\,,\ \ i \in S^{\pm} \tag{18}$$

for particles at $S^{\pm}$ boundaries corresponding to $z = \pm 1$.

Equations (17) can also be obtained from the extremum principle (6) if we replace the integral in it by the sum of energies in the channels and then write the extremum condition.

Some of the particles can be tangent to the $z = \pm 1$ boundaries, where the Dirichlet condition is posed, whereas part of them lie near these boundaries (Fig. 1). For the latter particles, the third boundary condition arises (for more details, see [10]).

The finite-dimensional problem (17), (18) is related to the input continuous problem. Under certain conditions, using its solution, one can construct estimates of the upper and lower bounds, which at a close packing of particles (formal definition is given in [3, 10]) join. Then the following statement will be proved.

Theorem. *Let the pair capacitances* $C_{ij}^{(2)}$ *have one order* of $f(\delta) \to \infty$ *when* $\delta \to 0$ *for all neighboring particles. Then the conductance* $A \to \infty$ *when* $\delta \to 0$ *is effective. The principal term A when* $\delta \to 0$ *is expressed in terms of* $\{t_i\}$ *— the solution of the discrete network problem (17), (18) in the form*

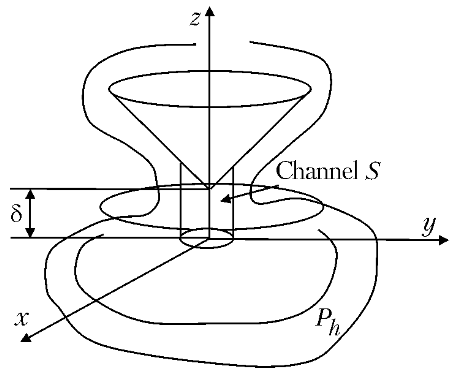

Fig. 3. Scheme of cone–plane pair for the three-dimensional case.

$$A = \frac{1}{4} \sum_{i,j=1}^{N} C_{ij}^{(2)} (t_i - t_j)^2 .$$

Results and Discussion. The field of application of the results obtained seems to be potentially very wide. Below, we give several main conclusions of the physical (technical) character.

Asymptotic Screening Effect. I. E. Tamm explains the asymptotic screening effect (without using the term "asymptotic") as follows [6, p. 53]: "if the dimensions of conductors are large compared to the spacing between them, ... the space between the capacitor plates will be protected, if not completely then to a considerable extent, by the plates themselves from the action of the external field." This reasoning is suited to a flat capacitor (where all problems are reduced to the boundary effect). For convex particles, as is seen from the appendices, the asymptotic screening effect may not be observed. The condition for its appearance is described in the screening lemma. And the physics of the phenomenon is as follows. If an unlimited increase in the capacitance is observed as the particles draw closer together, then it is accompanied by the effect of energy channeling (by virtue of item 1) of the lemma on screening, the increasing energy can concentrate only in the channel between particles). The strong field in this narrow channel will be largely determined by the local geometry of the channel and, by virtue of this, will be independent of the action of external fields created by other bodies. The particles themselves cannot protect the space between them, e.g., by two spheres. Note that for spheres the condition of item (2) of the screening lemma is met and the asymptotic screening effect takes place, as opposed to two cones with close vertices, although these bodies are fairly similar.

Influence of the Problem Dimensionality and the Shape of Particles on the Screening Effect. The above reasoning ignores the problem dimensionality; however, pair capacitances depend on it. The pair capacitances of smooth particles satisfy, as a rule, the condition of item (2) of the screening lemma for both dimensionality two and dimensionality three. But the pair capacitances of angular particles behave differently. We give corresponding examples.

Example 1. *Boundedness of the capacitance* $C_{\text{c-p}}^{(2)}$ *in* R^3 *for the cone–plane pair.* Consider a pair of particles having in the region of approach the cone–plane form (Fig. 3). The estimate of the upper bound for the capacitance of the two particles follows from (6) and is of the form

$$C_{\text{c-p}}^{(2)} \le \frac{1}{2} \int_{R^3 \setminus (S \cup \Pi)} |\nabla\varphi|^2 \, d\mathbf{x} , \quad \varphi(\mathbf{x}) = 1/2 \quad \text{on} \quad S , \quad \varphi(\mathbf{x}) = -1/2 \quad \text{on} \quad \Pi . \tag{19}$$

The end position of the particles at $\delta = 0$ permits introducing a channel S and a neighborhood P_h, in which one can define a test function that vanishes and has uniformly δ-bounded derivatives. Thus, to verify the capacitance boundedness, it is enough to make sure of the boundedness in the channel of the right side of (19) for some test function from V, i.e., one has to consider the integral

$$C_{\text{c-p}}^{(2)} \le \frac{1}{2} \int_{S} |\nabla\varphi|^2 \, d\mathbf{x} , \quad \varphi(\mathbf{x}) = 1/2 \quad \text{on} \quad S , \quad \varphi(\mathbf{x}) = -1/2 \quad \text{on} \quad \Pi . \tag{20}$$

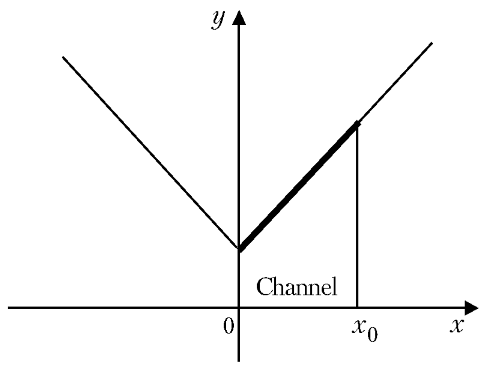

Fig. 4. Scheme of angle–plane pair for the two-dimensional case.

We assume $\varphi(\mathbf{x})$ to be linear in z:

$$\varphi(\mathbf{x}) = -1/2 + \frac{z}{\delta + \sqrt{x^2 + y^2}}$$

(we consider a circular cone with a vertex angle of $90^{\rm o}$). The function $\varphi(\mathbf{x})$ depends on all variables:

$$\frac{\partial\varphi}{\partial x}(\mathbf{x}) = \frac{\dfrac{xz}{\sqrt{x^2+y^2}}}{\left(\delta + \sqrt{x^2+y^2}\right)^2} = \frac{\dfrac{zr\cos\theta}{r}}{(\delta + r)^2} = \frac{z\cos\theta}{(\delta+r)^2},$$

$$\frac{\partial\varphi}{\partial y}(\mathbf{x}) = \frac{\dfrac{yz}{\sqrt{x^2+y^2}}}{\left(\delta + \sqrt{x^2+y^2}\right)^2} = \frac{z\sin\theta}{(\delta+r)^2},$$

$$\frac{\partial\varphi}{\partial z}(\mathbf{x}) = \frac{1}{\delta + \sqrt{x^2+y^2}} = \frac{1}{\delta + r}.$$

Calculate $\int_S |\nabla\varphi|^2 d\mathbf{x}$ for the above function. In cylindrical coordinates we have

$$\int_S \left(\frac{\partial\varphi}{\partial x}(\mathbf{x})\right)^2 d\mathbf{x} = \int_0^R \int_0^{r+\delta} \left(\frac{\partial\varphi}{\partial x}(\mathbf{x})\right)^2 r dr dz \le \int_0^R \frac{rdr}{(\delta+r)^4} \int_0^{r+\delta} z^2\, dz = \int_0^R \frac{rdr}{(\delta+r)^4}\frac{(\delta+r)^3}{3} \le \frac{R}{3},$$

which takes into account that $r < r + \delta$. The concrete value of R is immaterial. It is essential that it can be chosen to be independent of δ. The integral of $\left(\frac{\partial\varphi}{\partial y}(\mathbf{x})\right)^2$ is estimated similarly. Consider

$$\int_S \left(\frac{\partial\varphi}{\partial z}(\mathbf{x})\right)^2 d\mathbf{x} = \int_0^R \frac{rdr}{(\delta+r)^2} \int_0^{r+\delta} dz = \int_0^R \frac{rdr}{(\delta+r)^2}(\delta + r) = \int_0^R \frac{rdr}{\delta+r} \le R,$$

which takes into account that $r < r + \delta$.

Consequently, the capacitance $C^{(2)}_{\text{c–p}}$ is uniformly δ-bounded (also when $\delta \to 0$).

Example 2. *Unboundedness of the capacitance* $C^{(2)}_{\text{a–p}}$ *in* R^2 *for the angle–plane pair.* Consider a pair of particles having in the region of approach the angle–plane form (Fig. 4). The estimate of the lower bound for the capacitance of the pair of particles follows from (8) and is of the form

$$C^{(2)}_{\text{a–p}} \geq -\frac{1}{2} \int\limits_{R^3 \setminus (K \cup \Pi)} |\mathbf{v}|^2 \, d\mathbf{x} \int\limits_{\partial K \cup \partial \Pi} \varphi^0 \, \mathbf{vn} d\mathbf{x} \,, \quad \operatorname{div} \mathbf{v} = 0 \,, \tag{21}$$

where $\varphi^0(\mathbf{x}) = 1$ for the angle and $\varphi^0(\mathbf{x}) = -1$ for the plane. For the 90^{o} angle, as in Fig. 4, we introduce the function

$$\mathbf{v} = \begin{cases} \dfrac{\mathbf{e}_2}{\delta + x} \,, & 0 < x < x_0 \,; \\ 0 \,, & 0 \geq x \quad \text{and} \quad x \geq x_0 \end{cases}$$

($\mathbf{v} \neq 0$ at $0 < x < x_0$ and $\operatorname{div} \mathbf{v} = 0$ throughout the region). The normal vectors for the angle (Fig. 4) are $\mathbf{n} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$ and $\mathbf{n} = (0, 1)$. The right side of (21) will take on the form

$$-\frac{1}{2} \int_0^{x_0} dx \int_0^{x+\delta} \frac{1}{(\delta + x)} \, dy - \int_0^{x_0} \left(1 + \frac{1}{\sqrt{2}} \sqrt{2} \right) \frac{1}{\delta + x} \, dx \,, \tag{22}$$

where the appearance of the factor $\sqrt{2}$ is due to the dependence $ds = \sqrt{dx^2 + dy^2} = \sqrt{2}\, ds$. Integrating (22), we obtain

$$\frac{x_0}{2} - \left(-\frac{1}{2} + 2 \right) \int_0^{x_0} \frac{dx}{\delta + x} = \frac{x_0}{2} - \frac{3}{2} \ln (x + \delta) \Bigg|_0^{x_0} = \frac{x_0}{2} - \frac{3}{2} \ln (x_0 + \delta) - \frac{3}{2} \ln \delta \to \infty \quad \text{when} \quad \delta \to 0 \,.$$

Consequently, the capacitance $C^{(2)}_{\text{a–p}} \to \infty$ when $\delta \to 0$.

The pair capacitance of smooth bodies tends to infinity as they draw closer together (when $\delta \to 0$), while the order in δ depends on the shape of the bodies.

Effective conductance of the high-contrast composite and shape of particles. The technologies using high-contrast composites are based on the following reasoning: the particles can be considered to be perfect conductors, and the spacing between them is small. Therefore, a large total flux through the composite can be expected. This statement holds if the condition of item (2) of the screening lemma is met, but in the general case it is invalid. This condition is fulfilled, as a rule, for smooth particles and violated for angular ones, i.e., filling a composite with smooth particles, one can, in principle, realize in it high conducting properties, which seems to be impossible with the use of angular particles. In practice, particles of both types are used as a filler. The obtaining of smooth particles is more expensive than that of angular ones. But the angularity of particles leads to the loss of the screening effect.

Network models. As is seen, the possibility of finite-dimensional modeling of continuous high-contrast problems is determined by the energy concentration in the channels rather than by the contrast proper and the closeness of particles to one another. It can be said that network models stem from the screening effect and are a "secondary" effect. In the absence of this effect (which can take place in the three-dimensional case), the use of network models is, generally speaking, inadmissible. The condition $C^{(2)}_{ij} \to \infty$ when $\delta \to 0$ is sufficient for using network models and can be used in practice (since pair capacitances in R^2 and R^3 can be counted [8] or estimated) to check the applicability of network models.

Order of pair capacitances. In the theorem, the order of pair capacitances $f(\delta)$ is present. Below, we give the values of the orders of pair capacitances for some pairs of particles:

1) sphere–sphere, $f(\delta) = \ln \delta$;
2) cone–plane, $f(\delta) = 1$;
3) disk–disk, $f(\delta) = \sqrt{\delta}$;
4) plane–plane (flat capacitor), $f(\delta) = \frac{1}{8}$.

For cases (1), (3), and (4), $f(\delta) \to 0$ when $\delta \to 0$, and for (2) the function $f(\delta)$ remains finite when $\delta \to 0$.

Concept of capacitance. In this paper, the presentation of material is oriented to the problem of electrostatics. However, problems of the form of (1)–(5) arise in the theory of heat conduction and in elasticity theory, where the concept of capacitance is usually not used. Apparently, this concept can also be introduced for problems where it can play an important role. As a capacitance of vector problems, the energy at given displacements and rotations of two absolutely hard particles serves.

CONCLUSIONS

1. The relation of the Maxwell–Keller problem on the electric field in a system of closely spaced bodies to I. E. Tamm's asymptotic screening effect has been shown.

2. A sufficient condition for the appearance of the asymptotic screening effect, which is also a sufficient condition for the extensive network (finite-dimensional) approximation for the input continuous problem, has been obtained.

3. The relationship between the network model parameters and the pair capacitances of bodies has been established, which has made it possible to solve the problem on the influence of the shape of bodies on the averaged characteristics of the medium. The results obtained are applicable to heat conduction, diffusion, magnetics problems and the like.

NOTATION

$C_{ij}^{(2)}$, capacitance of sets D_i and D_j in R^n (n = 2, 3); $C^{S_{ij}}$, capacitance of sets D_i and D_j to the S_{ij} set; $\mathbf{e}_i$ (i = 1, 2, 3), unit vectors of the coordinate system; H^1, V, W, functional spaces; $I(\varphi)$, $J(\varphi)$, functionals; L, size of the region in the direction of the y-axis; $\mathbf{n}$, normal to the region boundary; r, θ, polar coordinates; $\mathbf{x} = (x, y, z)$, Cartesian rectangular coordinates; δ, characteristic spacing between neighboring bodies; φ, potential. Subscripts: a-p, angle–plane; c-p, cone–plane.

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